FIAN/TD/16-07
Theor. Math. Phys. 153 (2007) 1351-1363

STAR PRODUCT ALGEBRAS OF TEST FUNCTIONS

M. A. SOLOVIEV

Abstract. We prove that the Gelfand-Shilov spaces S^β_α are topological algebras under the Moyal $\star$-product if and only if $\alpha \geq \beta$. These spaces of test functions can be used to construct a noncommutative quantum field theory. The star product depends continuously on the noncommutativity parameter in their topology. We also prove that the series expansion of the Moyal product is absolutely convergent in S^β_α if and only if $\beta < 1/2$.

1. Introduction

In recent years, considerable attention has been given to noncommutative quantum field theories (QFTs), which occupy an intermediate position between the usual QFT and string theory (see, e.g., [1] for a review). The interaction terms in the Lagrangians of these theories are expressed in terms of a star product, which is a noncommutative and nonlocal deformation of the ordinary pointwise product of fields. This deformation leads to the loss of the commutativity of space-time coordinates and to a commutation relation of the form

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is a real antisymmetric matrix, constant in the simplest case.

The conceptual framework of quantum physics on a noncommutative space-time is still not conclusively established, and a serious effort is being made to clarify the questions of the causality of observables and of the implementation of symmetries, and also of the conditions of unitarity. In parallel with the study of actual models, there have also been attempts [2, 3, 4] to extend the axiomatic approach [5, 6] to noncommutative QFT. The description of quantum fields in terms of operator-valued distributions is one of the cornerstones of the axiomatic approach, and this raises the question of the optimal choice of test functions in noncommutative QFT. The relevance of such a question to finding solutions of QFTs was discussed in [7]. There is some evidence that the Schwartz space S used in the standard formalism [5, 6] is not quite adequate for a QFT on noncommutative space-time. As noted in [2], the tempered character of the Schwartz distributions can be incompatible with severe singularities caused by the UV/IR mixing intrinsic in noncommutative field theories. A further indication is an exponential growth of the correlation functions of some gauge-invariant operators in momentum space, found in [8, 9]. Moreover, the very structure of the star product, which is defined by an infinite-order differential operator, suggests that analytic test functions are best suited for use in a noncommutative QFT. Some subtleties in the

2000 *Mathematics Subject Classification.* 53D55, 46E25, 46F05, 81T05, 46L65.
Key words and phrases. noncommutative quantum field theory, Moyal star product, topological $*$-algebra, Gelfand-Shilov space.

derivation of the CPT and spin-statistics theorems in the enlarged formalism with analytic test functions were discussed in [10]. Here, we take a different approach to the question and propose a criterion for choosing a suitable test function space; this criterion implies that this space must be an algebra under the star product. The analysis is performed in the framework of Gelfand-Shilov spaces S^β_α, which are subalgebras of the Schwartz space with respect to the ordinary product. The index α determines the behavior of the test functions at infinity, and β determines their smoothness. The smaller these indices are, the smaller the space S^β_α and the larger its dual space of generalized functions. The Schwartz space S is the formal limit of S^β_α as $\alpha, \beta \to +\infty$.

In Sec. 2, we propose a simple way to analyze two well-known associative noncommutative products on the Schwartz space $S(\mathbb{R}^d)$. Both these products are generated by a Poisson structure on $\mathbb{R}^d$. The first product is a noncommutative deformation of the ordinary pointwise product of functions; its formal power series expansion in the noncommutativity parameter reproduces the Weyl-Groenewold-Moyal star product (which is hereafter called the Moyal product, as in most papers on this subject). We give an example that clearly demonstrates that this expansion does not converge in general in the topology of the Schwartz space. The second product, called the twisted convolution, is obtained from the first product by the Fourier transformation. In Sec. 3, we show that the proposed approach is also applicable to other spaces of test functions. Specifically, it allows proving that the subspaces $S^\beta_\alpha \subset S$ remain subalgebras of the Schwartz space under the noncommutative deformation if and only if $\alpha \geq \beta$. In Sec. 4, we study the conditions under which the series defining the Moyal $\star$-product converges, and we show that these conditions result in additional restrictions on the index β. In Sec. 5, we prove that for any $\star$-algebra S^β_α, the star product depends on the noncommutativity parameter θ continuously (i.e., this product is indeed a deformation of the ordinary product). We briefly discuss the obtained results in Sec. 6. In the appendix, we prove an elementary lemma which shows that the spaces under consideration contain functions with certain properties useful in analyzing the operation of star multiplication in these spaces and in finding the conditions for the convergence of the power series expansion in θ of the star product.

2. Star-product structure on the Schwartz space

Let f and g be smooth complex-valued functions on $\mathbb{R}^d$, and let $\theta^{\mu\nu}$ be a constant antisymmetric (possibly degenerate) $d \times d$ matrix. Then the Moyal $\star_\theta$-product of f and g is defined by the formula

$$\begin{aligned}(f \star_\theta g)(x) &= f(x) e^{i(\overleftarrow{\partial_\mu}\,\theta^{\mu\nu}\,\overrightarrow{\partial_\nu})/2} g(x) = \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!}\,\theta^{\mu_1\nu_1}\dots\theta^{\mu_n\nu_n}\partial_{\mu_1}\dots\partial_{\mu_n} f(x)\partial_{\nu_1}\dots\partial_{\nu_n} g(x) \quad (2)\end{aligned}$$

(summation over the indices $\mu_1, \dots \mu_n$ and $\nu_1, \dots \nu_n$ is implied), which is usually understood as a formal power series in θ. Product (2) reduces to the ordinary pointwise product of functions as $\theta \to 0$. The order-θ part coincides with $(i/2)\{f, g\}$, where the Poisson bracket $\{\cdot, \cdot\}$ is determined by the matrix $\theta^{\mu\nu}$. In particular,

$$x^\mu \star_\theta x^\nu - x^\nu \star_\theta x^\mu = i\theta^{\mu\nu}$$

and we obtain commutation relation (1). Thus, θ plays the role of a noncommutativity parameter. Because this parameter is the same throughout the paper, we write $\star$ instead of $\star_\theta$ in what follows.

We now suppose that the functions f and g decrease rapidly at infinity and belong to the Schwartz space $S(\mathbb{R}^d)$. Then each term in series expansion (2) has a Fourier transform,[1] which is readily calculated from the formulas $(\widehat{\partial_\mu f})(p) = ip_\mu \hat{f}(p)$ and $(\widehat{fg})(p) = (2\pi)^{-d}\int \hat{f}(q)\hat{g}(p-q)dq$. Summing over n, we obtain

$$(2\pi)^{-d}\int \hat{f}(q)\hat{g}(p-q)e^{-(i/2)\theta^{\mu\nu}q_\mu(p_\nu-q_\nu)}dq = (2\pi)^{-d}\int \hat{f}(q)\hat{g}(p-q)e^{(i/2)\theta^{\mu\nu}p_\mu q_\nu}dq. \quad (3)$$

In what follows, we use the index-free notation (whenever possible). The Poisson tensor $\theta^{\mu\nu}$ determines the antisymmetric bilinear form $\theta^{\mu\nu}p_\mu q_\nu$ on $\mathbb{R}^{d\prime}\times\mathbb{R}^{d\prime}$ and can be identified with the operator $\theta\colon \mathbb{R}^{d\prime}\to\mathbb{R}^d$ which takes each element $p\in\mathbb{R}^{d\prime}$ with the coordinates p_ν to a vector with the coordinates $(1/2)\theta^{\mu\nu}p_\nu$. (The coefficient $1/2$ is inserted to simplify the formulas that follow.) The function

$$(\hat{f}\circledast\hat{g})(p) \stackrel{\text{def}}{=} \int \hat{f}(q)\hat{g}(p-q)e^{i\langle p\,,\,\theta q\rangle}dq \quad (4)$$

is called the twisted convolution of $\hat{f}$ and $\hat{g}$. We let τ_p denote the shift operator $\hat{f}(q)\to\hat{f}(q-p)$ and τ_- the reflection $\hat{f}(q)\to\hat{f}(-q)$. Then the twisted convolution is obtained from the ordinary convolution $\int \hat{f}(q)(\tau_p\tau_-\hat{g})(q)\,dq$ by replacing τ_p with the operators $e^{i\langle p\,,\,\theta q\rangle}\tau_p$, which implement a projective representation of the translation group. Since $\hat{f},\hat{g}\in S=\mathcal{F}[S]$, it can be easily seen that function (4) is smooth and rapidly decreasing, i. e., it also belongs to the Schwartz space. The binary operation $(\hat{f},\hat{g})\to\hat{f}\circledast\hat{g}$ is associative. Indeed, we have

$$\begin{aligned}((\hat{f}\circledast\hat{g})\circledast\hat{h})(p) &= \int\left\{\int \hat{f}(k)\hat{g}(q-k)e^{i\langle q\,,\,\theta k\rangle}dk\right\}\hat{h}(p-q)e^{i\langle p\,,\,\theta q\rangle}dq\\ &= \int \hat{f}(k)\left\{\int \hat{g}(q)\hat{h}(p-k-q)e^{i\langle p-k\,,\,\theta q\rangle}dq\right\}e^{i\langle p\,,\,\theta k\rangle}dk = (\hat{f}\circledast(\hat{g}\circledast\hat{h}))(p).\end{aligned}$$

Obviously, $(\hat{f}\circledast\hat{g})^* = \hat{g}^*\circledast\hat{f}^*$. Therefore $(S,\circledast)$ is an involutive algebra with the complex conjugation as involution.

We let $f\times g$ denote the element of S whose Fourier transform is function (3), i.e.,

$$\widehat{f\times g} = (2\pi)^{-d}\,\hat{f}\circledast\hat{g}, \qquad f,g\in S(\mathbb{R}^d). \quad (5)$$

More explicitly,

$$(f\times g)(x) = \frac{1}{(2\pi)^{2d}}\int\int \hat{f}(q)\hat{g}(p)\,e^{i\langle q\,,\,x\rangle+i\langle p\,,\,x\rangle-i\langle q\,,\,\theta p\rangle}dqdp. \quad (6)$$

This function is called the twisted product of f and g. If the matrix θ is invertible and the Poisson structure is hence symplectic, then

$$(f\times g)(x) = \frac{1}{(2\pi)^d|\det\theta|}\int\int f(y)g(z)\,e^{i\left\langle\theta^{-1}(x-y)\,,\,x-z\right\rangle}dydz. \quad (7)$$

It is easily seen that nonlocal product (7) is translation and symplectic equivariant,[2] as is the ordinary pointwise product. Applying the inverse Fourier transformation to the power series expansion in θ of $(2\pi)^{-d}\,\hat{f}\circledast\hat{g}$, we obtain precisely initial series (2). But

[1] We use the definition of the Fourier operator $(\mathcal{F}f)(p) = \hat{f}(p) = \int f(x)e^{-i\langle p\,,\,x\rangle}dx$. The bracket $\langle\cdot\,,\cdot\rangle$ denoting pairing of the space $\mathbb{R}^d$ and its dual $\mathbb{R}^{d\prime}$ is identified with the standard Euclidean structure on $\mathbb{R}^d$.

[2] It can be shown that (7) follows from these properties combined with associativity and nonlocality.

it cannot be asserted that this series converges to $f \times g$ in the Schwartz space whose topology is determined by the norms[3]

$$\|f\|_N = \sup_{x\in\mathbb{R}^d} \sup_{|\kappa|\le N} (1+|x|)^N |\partial^\kappa f(x)|. \tag{8}$$

This is evident from the simplest example of Gaussian functions.

Proposition 1. *Let $d = 2$ and suppose that $\theta^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Let $f(x) = e^{-\gamma|x|^2}$, where $\gamma > 1$. Then the series expansion of $f \star f$ given by* (2) *does not converges in the topology of $S(\mathbb{R}^2)$.*

Proof. We consider the linear functional u on $S(\mathbb{R}^2)$, defined by $u(f) = \int f(0, x_2)\, dx_2$. Clearly, it is continuous in the topology of $S(\mathbb{R}^2)$, because $|u(f)| \le C\|f\|_2$, where $C = \int (1+|x_2|)^{-2} dx_2$. Let the terms in series (2) be denoted h_n. Then

$$u(h_n) = \int h_n(0, x_2) dx_2 = \frac{1}{2\pi} \int \hat{h}_n(p_1, 0) dp_1,$$

where

$$\hat{h}_n(p) = \frac{i^n}{(2\pi)^2 n!} \int \hat{f}(q) \hat{g}(p-q) \langle p\,, \theta q\rangle^n\, dq.$$

We show that if $\hat{f}(p) = \hat{g}(p) = (\pi/\gamma) e^{-|p|^2/4\gamma}$ and $\gamma > 1$, then $u(h_n) \nrightarrow 0$ as $n \to \infty$. Taking into account that $\langle p\,, \theta q\rangle = (p_1 q_2 - p_2 q_1)/2$ in this case, we obtain

$$u(h_n) = \frac{1}{8\pi\gamma^2} \left(\frac{i}{2}\right)^n \frac{1}{n!} \int \left\{ \int e^{-(q_1^2 + (p_1 - q_1)^2)/4\gamma} dq_1 \right\} p_1^n\, dp_1 \int e^{-q_2^2/2\gamma} q_2^n\, dq_2. \tag{9}$$

Let n be even. An elementary calculation gives

$$|u(h_n)| = \sqrt{\frac{\pi}{2\gamma}}\, \frac{\gamma^n}{n!} [1 \cdot 3 \dots (2n-1)]^2 \ge \sqrt{\frac{\pi}{2\gamma}}\, \frac{\gamma^n}{n}.$$

This proves the proposition. □

Because the Fourier transformation is an automorphism of the Schwartz space, this space also is an involutive algebra under the twisted product $\times$. Moreover, both the algebras $(S, \circledast)$ and $(S, \times)$ are topological. This can be verified by a straightforward estimation with norms (8) (see, e.g., [11]). A different way is based on using formula (6), which shows that the map $(f, g) \to f \times g$ is representable as the composition of five maps

$$S(\mathbb{R}^d) \times S(\mathbb{R}^d) \xrightarrow{\otimes} S(\mathbb{R}^{2d}) \xrightarrow{\mathcal{F}} S(\mathbb{R}^{2d}) \xrightarrow{\cdot e^{-i(q,\theta p)}} S(\mathbb{R}^{2d}) \xrightarrow{\mathcal{F}^{-1}} S(\mathbb{R}^{2d}) \xrightarrow{\widehat{\mathsf{m}}} S(\mathbb{R}^d), \tag{10}$$

where $\widehat{\mathsf{m}}$ denotes the restriction to the diagonal $h(x, y) \to h(x, x)$. By Schwartz's kernel theorem, the space $S(\mathbb{R}^{2d})$ coincides with the completion of the tensor product $S(\mathbb{R}^d) \otimes_\pi S(\mathbb{R}^d)$ endowed with the projective topology. The map $(f, g) \to f \otimes g$ is continuous in this topology. Furthermore, there is a one-to-one correspondence between the set of continuous bilinear maps $S(\mathbb{R}^d) \times S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ and the set of continuous linear maps $S(\mathbb{R}^{2d}) \to S(\mathbb{R}^d)$. In particular, the linear map $\widehat{\mathsf{m}}$ is associated with the ordinary multiplication $\mathsf{m}\colon (f, g) \to f \cdot g$, and the continuity of $\widehat{\mathsf{m}}$ follows from (and amounts to) the fact that $S(\mathbb{R}^d)$ is a topological algebra under ordinary multiplication. The Fourier transformation is not only linear but also a topological automorphism of S, and the function $e^{-i(q,\theta p)}$ is obviously an multiplier for this space, i.e., the multiplication by this function maps S into itself continuously. Therefore, all the maps involved

[3]In (8), $\kappa \in \mathbb{Z}_+^d$, and the notation $|\kappa| = \kappa_1 + \dots + \kappa_d$, $\partial^\kappa = \partial^{|\kappa|}/(\partial x_1^{\kappa_1} \cdots \partial x_d^{\kappa_d})$ is used.

in composition (10) are continuous; hence the algebras $(S, \circledast)$ and $(S, \times)$ are indeed topological.

Representation (10) also allows finding subalgebras of $(S, \circledast)$ and $(S, \times)$ that become complete topological algebras when they are endowed with an appropriate topology.

3. The algebras $(S^\beta_\alpha, \times)$ and $(S^\alpha_\beta, \circledast)$

We recall the definition and basic properties of the S-type spaces introduced by Gelfand and Shilov [12]. The space $S^\beta_\alpha(\mathbb{R}^d)$, where $\alpha, \beta \geq 0$, consists of the functions $f \in S$ satisfying the inequalities[4]

$$|\partial^\kappa f(x)| \leq C B^{|\kappa|} \kappa^{\beta\kappa} e^{-|x/A|^{1/\alpha}}, \tag{11}$$

where the constants C, A, and B depend on f and the conventional multi-index notation is used, in particular, $\kappa^{\beta\kappa} = \kappa_1^{\beta\kappa_1} \dots \kappa_d^{\beta\kappa_d}$. (If $\kappa_i = 0$, then $\kappa_i^{\beta\kappa_i}$ is taken to be 1). We also write S^β_α for this space when this cannot lead to confusion. It is the union of the Banach spaces $S^{\beta,B}_{\alpha,A}$ with the norms

$$\|f\|_{A,B} = \sup_{x,\kappa} e^{|x/A|^{1/\alpha}} \left| \frac{\partial^\kappa f(x)}{B^{|\kappa|} \kappa^{\kappa\beta}} \right|. \tag{12}$$

and is endowed with the inductive limit topology by the natural maps $S^{\beta,B}_{\alpha,A} \to S^\beta_\alpha$. The space S^β_α is nontrivial if $\alpha + \beta \geq 1$ with the exceptional cases $\alpha = 0$ and $\beta = 0$, where the nontriviality conditions are the respective strict inequalities $\beta > 1$ and $\alpha > 1$. As shown in [12], the connecting maps $S^{\beta,B}_{\alpha,A} \to S^{\beta,B'}_{\alpha,A'}$, $A' > A$, $B' > B$, are compact. Hence, S^β_α is a complete Montel (perfect) space. The Fourier transformation is a linear topological isomorphism of S^β_α onto S^α_β. Every nontrivial space of type S is a topological algebra under ordinary multiplication as well as under ordinary convolution.

Theorem 1. *If $\alpha \geq \beta$, then $S^\beta_\alpha(\mathbb{R}^d)$ is a topological algebra under twisted product* (7) *and $S^\alpha_\beta(\mathbb{R}^d) = \mathcal{F}[S^\beta_\alpha(\mathbb{R}^d)]$ is a topological algebra under twisted convolution* (4).

Proof. By (5), the second statement in the theorem is equivalent to the first. As Mityagin showed [13], the spaces of type S are nuclear and

$$S^\beta_\alpha(\mathbb{R}^d) \mathbin{\widehat{\otimes}_\pi} S^\beta_\alpha(\mathbb{R}^d) = S^\beta_\alpha(\mathbb{R}^{2d}),$$

where the hat on $\otimes$ denotes completion. It therefore suffices to prove that the function $e^{-i\langle q\,,\,\theta p\rangle}$ is a multiplier for $S^\alpha_\beta(\mathbb{R}^{2d})$ under the indicated restriction on the indices α and β. Then the operation $(f, g) \to f \times g$ is representable as the composition of continuous maps

$$S^\beta_\alpha(\mathbb{R}^d) \times S^\beta_\alpha(\mathbb{R}^d) \xrightarrow{\otimes} S^\beta_\alpha(\mathbb{R}^{2d}) \xrightarrow{\mathcal{F}} S^\alpha_\beta(\mathbb{R}^{2d}) \xrightarrow{\cdot e^{-i(q,\theta p)}} S^\alpha_\beta(\mathbb{R}^{2d}) \xrightarrow{\mathcal{F}^{-1}} S^\beta_\alpha(\mathbb{R}^{2d}) \xrightarrow{\widehat{\mathsf{m}}} S^\beta_\alpha(\mathbb{R}^d), \tag{13}$$

in complete analogy with the case of the Schwartz space considered in the preceding section.

According to [12], a function $\chi(s)$ is a multiplier of S^α_β if it satisfies the estimate

$$|\partial^\kappa \chi(s)| \leq C_\epsilon A_\epsilon^{|\kappa|} \kappa^{\alpha\kappa} \exp\{\,|\epsilon s|^{1/\beta}\} \tag{14}$$

[4] For $\alpha = 0$, the exponential in (11) should be replaced with the characteristic function of the set $|x| \leq A$.

for any $\epsilon > 0$. Here, we have an entire function of $2d$ variables and the required estimate can easily be derived from the Cauchy inequality

$$|\partial^\kappa \chi(s)| \leq \kappa!\, r^{-|\kappa|} \sup_{w\in D_r} |\chi(s-w)|, \tag{15}$$

where $D_r = \{w \in \mathbb{C}^{2d} \colon |w_j| < r\, \forall j\}$. We set $s = (p, q)$ and $w = (u, v)$ and use the notation $|\theta| = \sum_{j,k} |\theta^{jk}|$. Then $|\operatorname{Im}\langle q - v\,, \theta(p-u)\rangle| \leq r\,|\theta|\,(|q| + |p| + 2r)$, and we obtain

$$|\partial^\kappa \exp\{-i\,\langle q\,, \theta p\rangle\}| \leq \kappa!\, r^{-|\kappa|} \exp\{\, r\,|\theta|\,(|s| + 2r)\}, \tag{16}$$

where $|s| = |p| + |q|$. Because $\kappa! \leq \kappa^\kappa$ and the radius r of the polydisk can be taken arbitrarily small, we immediately conclude that the function $e^{-i\langle q\,,\theta p\rangle}$ is a multiplier for S^1_1 and also for any space S^α_β with the indices satisfying $\alpha \geq 1$ and $\beta \leq 1$. In particular, this is the case for all S^α_0 because they are nontrivial only if $\alpha > 1$.

If $\beta > 1$, then we take

$$r = \frac{1}{|\theta||s|} |\epsilon s|^{1/\beta}. \tag{17}$$

This expression tends to zero as $|s| \to \infty$. If r is chosen thus, then the exponential in the right-hand side of (16) does not exceed $Ce^{|\epsilon s|^{1/\beta}}$ everywhere in the region $|s| \geq 1$. Furthermore, we have

$$\frac{\kappa!}{r^\kappa} \leq A^{|\kappa|}\kappa^{\beta\kappa} \sup_\kappa \frac{1}{(Ar)^{|\kappa|}\kappa^{(\beta-1)\kappa}} \leq A^{|\kappa|}\kappa^{\beta\kappa} e^{(2d\beta/e)|Ar|^{-1/(\beta-1)}}.$$

Substituting (17), we see that the last exponential is also dominated by $e^{|\epsilon s|^{1/\beta}}$ if A is sufficiently large. Consequently, the function under consideration is a multiplier for S^β_β, $\beta > 1$, as well as for all S^α_β whose indices satisfy the inequalities $1 < \beta \leq \alpha$.

In the case $1/2 \leq \beta < 1$, we use the Young inequality

$$ab \leq \beta a^{1/\beta} + (1-\beta) b^{1/(1-\beta)}, \qquad a, b \geq 0, \tag{18}$$

setting $a = |\epsilon s|$ and $b = r|\theta|/\epsilon$. Choosing $r = |\kappa|^{1-\beta}$, we find that the right-hand side of (16) is dominated by the expression $A_\epsilon^{|\kappa|}\kappa^{\beta\kappa}e^{\,|\epsilon s|^{1/\beta}}$. Therefore, in this case, $e^{-i\langle q\,,\theta p\rangle}$ is a multiplier for S^β_β and for S^α_β, where $\alpha > \beta$. Finally, if $0 < \beta < 1/2$, then we again use (18), but we now take $r = |\kappa|^\beta$ and conclude that $e^{-i\langle q\,,\theta p\rangle}$ is a multiplier for $S^{1-\beta}_\beta$. This completes the proof because the spaces S^α_β are trivial if $\alpha < 1 - \beta$. □

We note that the Fourier-invariant spaces S^β_β are topological algebras under both of the operations $\times$ and $\circledast$. The space $S^{1/2}_{1/2}$ is smallest of these and plays a special role.

We now show that the restrictions imposed by Theorem 1 on the indices of the spaces of type S are necessary for these spaces to be star-product algebras.

Theorem 2. *Let the $\times$-product be determined by a nondegenerate matrix $\theta^{\mu\nu}$. If $\alpha < \beta$ and the space $S^\beta_\alpha(\mathbb{R}^d)$ is nontrivial, then this space contains functions f and g such that $f \times g \notin S^\beta_\alpha(\mathbb{R}^d)$.*

Proof. Because the matrix $\theta^{\mu\nu}$ is antisymmetric, definition (7) can be rewritten as

$$(f \times g)(x) = \frac{1}{(2\pi)^d |\det\theta|} \int\int f(y)g(z)e^{i\left\langle \theta^{-1}(z-y\,,\,x\right\rangle + i\left\langle\theta^{-1}y\,,\,z\right\rangle} dydz. \tag{19}$$

We first consider the simplest case $\alpha = 0$. All elements of S^β_0 are compactly supported. It is evident from (19) that the $\times$-product of such functions admits an analytic continuation to $\mathbb{C}^d$. But nontrivial analytic functions cannot have compact support, and we conclude that the product $f \times g$ of two elements of S^β_0 belongs to the same space only

if $(f \times g)(x) \equiv 0$. But we can easily find functions $f, g \in S_0^\beta$ such that $(f \times g)(0) > 0$. Indeed, we have

$$(f \times g)(0) = \frac{1}{|\det\theta|} \int f(y)\hat{g}(-\theta^{-1}y)dy. \tag{20}$$

Because S_0^β and S_β^0 are algebras under the ordinary multiplication, we can construct nonnegative functions $f \in S_0^\beta$ and $\hat{g} \in S_\beta^0$ starting from any nontrivial elements of these spaces. Furthermore, we can make the integrand in (20) nonvanishing by using the translation invariance of S_0^β. Then $f \times g \notin S_0^\beta$.

We now suppose that $0 < \alpha < \beta < 1$. We take $f, g \in S_{1-\beta}^\beta$ such that $(f \times g)(0) > 0$. These functions decrease no worse than exponentially of order $1/(1-\beta)$ with a finite type. Using Young inequality (18) with $b = |\theta^{-1}(z-y)|/B$ and $a = B|\operatorname{Im} x|$, where B is sufficiently large, we deduce that $(f \times g)(x)$ can be analytically continued to $\mathbb{C}^d$ as an entire function of an order $\le 1/\beta$. We consider the analytic continuation in the variable x^1 for $x^2 = \cdots = x^d = 0$. It is well known that any nontrivial entire function of finite order of growth cannot have an exponential decrease of a greater order along a direction of the complex plane. (This is an immediate consequence of Theorem 2.5.4 in [14].) Therefore the inequality $1/\alpha > 1/\beta$ implies that $f \times g \notin S_\alpha^\beta$. If $0 < \alpha < 1$ and $\beta > 1$, then we obtain the same conclusion taking functions f and g in $S_{1-\beta'}^{\beta'}$, where $\alpha < \beta' < 1$.

Let $\alpha = 1$ and $\beta > \alpha$. We again take functions f and g in S_0^β such that $f \times g \not\equiv 0$. Then the analytic continuation of the product $f \times g$ is an entire function of order 1 and finite type. By the Paley-Wiener theorem, the support of $\widehat{f \times g}$ is compact. We can also demonstrate this by shifting the plane of integration in representation (4). By the Cauchy-Poincaré theorem, this leaves the integral unchanged because of the analyticity and rapid degrease of the elements of S_β^0 at the real infinity. Namely, for any $u \in \mathbb{R}^d$, we have the estimates

$$|\hat{f}(q + iu)| \le C\, e^{-|q/B|^{1/\beta} + r|u|}, \qquad |\hat{g}(p - q - iu)| \le C'\, e^{r'|u|}$$

and hence

$$(\hat{f} \circledast \hat{g})(p) = \int \hat{f}(q+iu)\hat{g}(p-q-iu)e^{i\langle p\,,\,\theta q\rangle - (p,\theta u)}dq.$$

Assuming for simplicity that $d = 2$ and $\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we obtain

$$|(\hat{f} \circledast \hat{g})(p)| \le C'' e^{(r+r')|u| - p_1 u_2 + p_2 u_1}. \tag{21}$$

The support of this convolution is therefore contained in the square $\max\{|p_1|, |p_2|\} \le r + r'$. Indeed, we assume that $p_1 > r + r'$, for example. If $u_1 = 0$ and $u_2 \to +\infty$, then the right-hand side of (21) vanishes. Because all elements of S_β^1 are analytic, we deduce that $f \times g \notin S_1^\beta$.

Considering the remaining case $\alpha > 1$, we again assume that $d = 2$ and $\theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This does not result in any loss of generality, because the spaces S_α^β are invariant under the linear changes of variables, and we can use a symplectic basis in $\mathbb{R}^d$. Let $f(x) = f_1(x_1)f_2(x_2)$ and $g(x) = g_1(x_1)g_2(x_2)$, where $f_i, g_i \in S_\alpha^\beta(\mathbb{R})$. A simple calculation gives

$$(f \times g)(x_1, 0) = \frac{1}{2\pi} \int \hat{f}_1(z_2)g_2(-z_2)e^{ix_1z_2}dz_2 \int f_2(y_2)\hat{g}_1(y_2)e^{ix_1y_2}dy_2. \tag{22}$$

We set $g_1(\xi) = f_1(\xi)$ and $g_2(\xi) = f_2(-\xi)$. Then

$$(f \times g)(x_1, 0) = h^2(x_1),$$

where $h = \mathcal{F}^{-1}(\hat{f}_1 f_2)$. Because $\alpha > 1$, the function f_1 can be chosen such that its Fourier transform $\hat{f}_1$ is identically equal to 1 in a neighborhood of zero. As shown in the appendix, the space $S^\beta_\alpha(\mathbb{R})$ contains a function whose successive derivatives are not less than $n^{\beta n}$ in absolute value. Let f_2 be such a function. Then

$$\partial^n(\hat{f}_1 f_2)(0) = \partial^n \hat{h}(0) \geq n^{\beta n}, \qquad n = 0, 1, 2, \dots. \tag{23}$$

It follows that $f \times g \notin S^\beta_\alpha$, because the function h would otherwise satisfy the inequality

$$|h(\xi)| \leq C e^{-|\frac{\xi}{A}|^{1/\alpha}} \tag{24}$$

with some constants $C, A > 0$, and we would then have

$$|\partial^n \hat{h}(0)| \leq C \int |\xi|^n e^{-|\xi/A|^{1/\alpha}} d\xi = 2CA^{n+1} \int_0^\infty t^n e^{-t^{1/\alpha}} dt$$
$$\leq C' A^n \max_{t>0} \left(t^n e^{-(1/2)t^{1/\alpha}}\right) = C' A^n (2\alpha n/e)^{\alpha n},$$

which contradicts inequality (23) for $\beta > \alpha$. Theorem 2 is thus proved. □

The following analogue of Proposition 1 holds.

Proposition 2. *Let $d = 2$, $\theta^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and $\alpha > \beta$. If $\beta \geq 1/2$, then there is a function $f \in S^\beta_\alpha(\mathbb{R}^2)$ such that the series expansion of $f \star f$ given by (2) does not converges in the topology of $S^\beta_\alpha(\mathbb{R}^2)$.*

Proof. We consider the same linear functional $u(f)$ as in the proof of Proposition 1. Clearly it is continuous in the topology of $S^\beta_\alpha(\mathbb{R}^2)$. Using the lemma proven in the appendix and taking into account that the space $S^\alpha_\beta(\mathbb{R})$ is dilation invariant, we see that under the condition $\beta \geq 1/2$, it contains a positive even function that dominates the Gaussian function $e^{-|s|^2/(4\gamma)}$. We also note that γ can be taken arbitrarily large. Let $\hat{f}(p_1, p_2)$ be the tensor product of such functions. We then have

$$\hat{f}(p) \geq e^{-|p|^2/(4\gamma)}.$$

The further arguments are similar to those used to prove Proposition 1, with the only difference that (9) is replaced with the estimate

$$|u(h_n)| \geq \frac{c}{2^n n!} \int \left\{ \int e^{-(q_1^2 + (p_1 - q_1)^2)/(4\gamma)} dq_1 \right\} p_1^n \, dp_1 \int e^{-2q_2^2/(2\gamma)} q_2^n \, dq_2,$$

which holds for every even integer n. □

4. Convergence of the $\star$-product

The next theorem establishes a simple sufficient condition for the pointwise convergence of the series obtained from (2) by the Fourier transformation.

Theorem 3. *If $f, g \in S^\beta_\alpha(\mathbb{R}^d)$, where $\beta < 1$, then the series*

$$\sum_n \frac{i^n}{n!} \int \hat{f}(q) \hat{g}(p - q) \langle p\,, \theta q \rangle^n \, dq \tag{25}$$

converges to the function $(\hat{f} \circledast \hat{g})(p)$ uniformly on every compact set $Q \subset \mathbb{R}^d$.

Proof. The condition $\beta < 1$ implies that

$$|\hat{f}(q)| \le C_a e^{-a|q|}, \qquad |\hat{g}(q)| \le C'_a e^{-a|q|} \tag{26}$$

for any $a > 0$. Let r be so large that Q is contained in the ball $|p| < r$, and let $a > 2r|\theta|$. Then for any $N = 0, 1, \dots$ and $R > 0$, the estimate

$$\left|\sum_{n=0}^{N} \frac{i^n}{n!} \int_{|q|>R} \hat{f}(q)\hat{g}(p-q)\langle p\,,\theta q\rangle^n \, dq - \int_{|q|>R} \hat{f}(q)\hat{g}(p-q) e^{i\langle p\,,\theta q\rangle} dq\right| \le$$

$$\le C'_a \int_{|q|>R} |\hat{f}(q)| \left(e^{|\langle p\,,\theta q\rangle|} + 1\right) \le$$

$$\le C_a C'_a \int_{|q|>R} e^{-2r|\theta||q|}(e^{r|\theta||q|} + 1) dq, \qquad p \in Q, \tag{27}$$

holds. We take R so large that the right-hand side of (27) is less than $\epsilon/2$. Next, we choose N_ϵ such that for $N > N_\epsilon$, the inequality

$$\sup_{|p|\le r; |q| \le R} \left|\sum_{n=0}^{N} \frac{i^n}{n!} \langle p\,,\theta q\rangle^n - e^{i\langle p\,,\theta q\rangle}\right| < \frac{\epsilon}{2 v_R C_a C'_a}$$

holds, where v_R is the volume of the ball $|q| < R$. Then we have

$$\sup_{p\in Q} \left|\sum_{n=0}^{N} \frac{i^n}{n!} \int \hat{f}(q)\hat{g}(p-q)\langle p\,,\theta q\rangle^n \, dq - (\hat{f} \circledast \hat{g})(p)\right| < \epsilon$$

for any $N > N_\epsilon$, which completes the proof. □

Theorem 4. *If $f, g \in S^\beta_\alpha(\mathbb{R}^d)$, where $\beta < 1/2$, then series (2) is absolutely summable in the space $S^\beta_\alpha(\mathbb{R}^d)$, and its sum is the function $f \times g$ defined by (7).*

Proof. We note that if $\beta < 1/2$ and the space S^β_α is nontrivial, then $\alpha > \beta$. As before, we let h_n denote the nth term in series (2). It suffices to show that this series is absolutely summable in the Banach space $S^{\beta,B}_{\alpha,A}$ if A and B are sufficiently large. In other words, the convergence of the number series $\sum_n \|h_n\|_{A,B}$ should be examined. Let $f \in S^{\beta,B_1}_{\alpha,A_1}$ and $g \in S^{\beta,B_2}_{\alpha,A_2}$. Then we have

$$|\partial^\kappa f(x)| \le \|f\|_{A_1,B_1} B_1^{|\kappa|} \kappa^{\beta\kappa} e^{-|x/A_1|^{1/\alpha}}, \quad |\partial^\kappa g(x)| \le \|g\|_{A_2,B_2} B_2^{|\kappa|} \kappa^{\beta\kappa} e^{-|x/A_2|^{1/\alpha}}. \tag{28}$$

We let μ and ν denote the multi-indices in $\mathbb{Z}^d_+$ that correspond to the n-tuples $(\mu_1, \dots, \mu_n)$ and $(\nu_1, \dots, \nu_n)$ involved in (2). These multi-indices are determined by the equations $\partial^\mu = \partial_{\mu_1} \dots \partial_{\mu_n}$, $\partial^\nu = \partial_{\nu_1} \dots \partial_{\nu_n}$. Clearly, $|\mu| = |\nu| = n$. Let A be so large that $A^{-1/\alpha} \le A_1^{-1/\alpha} + A_2^{-1/\alpha}$, and let $C = \|f\|_{A_1,B_1}\|g\|_{A_2,B_2}$. Using Leibniz's formula and the elementary inequalities $(l+m)^{l+m} \le e^{l+m} l^l m^m$ and $l^l m^m \le (l+m)^{l+m}$, we obtain

$$e^{|x/A|^{1/\alpha}} |\partial^\kappa(\partial^\mu f \partial^\nu g)(x)| \le$$

$$\le C \sum_\lambda \binom{\kappa}{\lambda} B_1^{|\kappa-\lambda+\mu|} B_2^{|\lambda+\nu|} (\kappa-\lambda+\mu)^{\beta(\kappa-\lambda+\mu)} (\lambda+\nu)^{\beta(\lambda+\nu)} \le$$

$$\le C B_1^{|\mu|} B_2^{|\nu|} e^{\beta|\kappa+\mu+\nu|} \mu^{\beta\mu} \nu^{\beta\nu} \sum_\lambda \binom{\kappa}{\lambda} B_1^{|\kappa-\lambda|} B_2^{|\lambda|} (\kappa-\lambda)^{\beta(\kappa-\lambda)} \lambda^{\beta\lambda} \le$$

$$\le C(B_1 B_2 e^{2\beta})^n n^{2\beta n} [e^\beta (B_1 + B_2)]^{|\kappa|} \kappa^{\beta\kappa}.$$

Taking $B \geq e^{\beta}(B_1 + B_2)$, we obtain the estimate

$$\|h_n\|_{A,B} \leq C(B_1 B_2 e^{2\beta}|\theta|)^n \, \frac{n^{2\beta n}}{n!}. \tag{29}$$

Using the inequality $n! \geq n^n/e^n$, we deduce that the series $\sum_n \|h_n\|_{A,B}$ is indeed convergent under the condition $\beta < 1/2$. Now, we take into account that the Fourier transformation is a topological isomorphism of $S^{\beta}_{\alpha}(\mathbb{R}^d)$ onto $S^{\alpha}_{\beta}(\mathbb{R}^d)$ and apply Theorem 3, which shows that the function $f \times g$ is the sum of absolutely summable series (2). Theorem 4 is thus proved. $\square$

Corollary 1. *The twisted product $\times$ on the Schwartz space $S(\mathbb{R}^d)$ (as well as on any space $S^{\beta}_{\alpha}(\mathbb{R}^d)$, where $\alpha \geq \beta \geq 1/2$) is a continuous extension of $\star$-product (2) of the topological algebras $S^{\beta'}_{\alpha}(\mathbb{R}^d)$, $\beta' < 1/2$, for which product (2) is well defined.*

Indeed, any nontrivial space $S^{\beta'}_{\alpha}(\mathbb{R}^d)$ is dense in $S(\mathbb{R}^d)$ and also in $S^{\beta}_{\alpha}(\mathbb{R}^d)$, where $\beta > \beta'$. Therefore, $(f, g) \to f \times g$ is a unique continuous map $S(\mathbb{R}^d) \times S(\mathbb{R}^d) \to S(\mathbb{R}^d)$ (and $S^{\beta}_{\alpha}(\mathbb{R}^d) \times S^{\beta}_{\alpha}(\mathbb{R}^d) \to S^{\beta}_{\alpha}(\mathbb{R}^d)$) coinciding with the map $(f, g) \to f \star g$ on $S^{\beta'}_{\alpha}(\mathbb{R}^d) \times S^{\beta'}_{\alpha}(\mathbb{R}^d)$.

5. Continuity of the deformation

We now show that if $\theta \to 0$, then the product $f \times_{\theta} g$ tends to the ordinary product $f \cdot g$ in the topology of the algebras containing these functions.

Theorem 5. *Let $f, g \in S^{\beta}_{\alpha}(\mathbb{R}^d)$, where $\alpha \geq \beta$. The product $f \times_{\theta} g$ depends continuously on the noncommutativity parameter θ.*

Proof. Decomposition (13) reduces the problem to verifying that the operator on $S^{\alpha}_{\beta}(\mathbb{R}^{2d})$ consisting in multiplication by $e^{-i(q,\theta p)}$ is continuous in the parameter θ. It suffices to show this for $\theta = 0$. We use the notation $s = (p, q)$ and $e_{\theta}(s) = e^{-i(q,\theta p)}$. The analysis performed in Sec. 3 shows that

$$|\partial^{\kappa}(1 - e_{\theta}(s))| \leq C_{\epsilon} A_{\epsilon}^{|\kappa|} \kappa^{\alpha\kappa} e^{|\epsilon s|^{1/\beta}} \tag{30}$$

for any $\epsilon > 0$ and this estimate is uniform in θ for $|\theta| \leq 1$. (If θ is bounded thus, then we can set $|\theta| = 1$ in (17).) Furthermore, using the Taylor series expansion, we see that $1 - e_{\theta}(s) = |\theta|\, \chi_{\theta}(s)$, where $|\chi_{\theta}(s)| \leq e^{|s|^2}$ for $|\theta| \leq 1$. To estimate the derivatives of the entire function χ_{θ}, we use formula (15), but we now take the radiuses r_j of the polydisk D_r to be $\sqrt{\kappa_j}$. Then we obtain

$$|\partial^{\kappa} \chi_{\theta}(s)| \leq \frac{\kappa!}{r^{\kappa}} e^{2r^2 + 2|s|^2} \leq e^{2|\kappa|} \kappa^{\kappa/2} e^{2|s|^2}.$$

(we use the inequality $k! \leq e^d \kappa^{\kappa}/2^{|\kappa|}$ in the last step). Because S^{α}_{β} is nontrivial only if $\alpha + \beta \geq 1$, the condition $\alpha \geq \beta$ implies that $\alpha \geq 1/2$. Therefore, we have the inequalities

$$|\partial^{\kappa}(1 - e_{\theta}(s))| \leq |\theta| e^{2|\kappa|} \kappa^{\alpha\kappa} e^{2|s|^2}. \tag{31}$$

in addition to (30). Let $h \in S^{\alpha, A}_{\beta, B}(\mathbb{R}^{2d})$. We show that there are constants $A' \geq A, B' \geq B$ such that $\|(1 - e_{\theta})h\|_{A',B'} \to 0$ as $|\theta| \to 0$. To simplify the formulas in what follows, we set $B = 1/3^{\beta}$ without loss of generality, because S^{α}_{β} is invariant under dilations and $e_{\theta}(\lambda s) = e_{\lambda^2\theta}(s)$. Then

$$|\partial^{\kappa} h(s)| \leq \|h\|_{A,B} A^{|\kappa|} \kappa^{\alpha\kappa} e^{-3|s|^{1/\beta}}. \tag{32}$$

Applying Leibniz's formula and using inequality (30) with $\epsilon = 1$ and inequalities (31) and (32), we obtain the two estimates

$$|\partial^\kappa[(1-e_\theta(s))h(s)]| \le \begin{cases} C_h(A+A_1)^{|\kappa|}\kappa^{\alpha\kappa}e^{-2|s|^{1/\beta}}, \\ |\theta|C'_h(A+e^2)^{|\kappa|}\kappa^{\alpha\kappa}e^{2|s|^2}. \end{cases}$$

Let $A' = A + \max(A_1, e^2)$ and $B' = 1$. Then we have

$$\sup_\kappa e^{|s/B'|^{1/\beta}} \frac{|\partial^\kappa[(1-e_\theta(s))h(s)]|}{A'^{|\kappa|}\kappa^{\alpha\kappa}} \le \begin{cases} C_h e^{-|R|^{1/\beta}}, & |s| \ge R, \\ |\theta|C'_h e^{2|R|^2+|R|^{1/\beta}}, & |s| < R. \end{cases}$$

Given $\delta > 0$, we choose R such that $C_h e^{-|R|^{1/\beta}} \le \delta$. Then $\|(1-e_\theta)h\|_{A',B'} \le \delta$ for $|\theta| \le (\delta/C'_h)e^{-2|R|^2-|R|^{1/\beta}}$. The theorem is proved. □

6. Conclusion

The performed analysis shows that the spaces of analytic test functions that were previously used to construct a quantum theory of nonlocal interactions [15, 16, 17] are topological algebras under the star product. This means that they can also be used in QFT on a noncommutative space-time along with the functional analytic methods developed in extending Wightman's axiomatic approach to nonlocal fields.

Some authors (see, e.g., [3, 4, 18]) considered a $\star$-product of field operators $\phi(x)$ at different space-time points, using the definition

$$\phi(x_1)\star\phi(x_2) = e^{(i/2)\,\theta^{\mu\nu}(\partial/\partial x_1^\mu)(\partial/\partial x_2^\nu)}\phi(x_1)\phi(x_2). \tag{33}$$

This definition can easily be extended to any finite number of operators at different points (formula (2.24) in [1]). The axiomatic formulation of noncommutative QFT proposed in [3] is based on the corresponding modification of the Wightman functions written as the vacuum expectation value

$$\langle 0|\phi(x_1)\star\phi(x_2)\star\cdots\star\phi(x_n)|0\rangle. \tag{34}$$

There is only one way to give a rigorous mathematical meaning to formal definitions (33) and (34). Namely, the infinite-order differential operator in (33) should be regarded as the dual of the operator $e^{(i/2)\,\theta^{\mu\nu}(\partial/\partial x_1^\mu)(\partial/\partial x_2^\nu)}$ acting on suitable test functions. Clearly, the latter operator is the Fourier transform of the multiplier $e^{-i\langle p_1,\theta p_2\rangle}$ and is well defined on the spaces $S^\beta_\alpha(\mathbb{R}^{2d})$ whose indices satisfy the restriction $\alpha \ge \beta$ established by Theorem 1. The arguments used to prove Theorem 4 show that under the stronger condition $\beta < 1/2$, the series expansion of this operator converges on every test function. Such test function spaces can be used as a natural initial domain of this operator with a possible further extension depending on the model under consideration.

In conclusion we note that in developing the Weyl-Wigner-Groenewold-Moyal approach to quantum mechanics, much attention was given to specifying those pairs of tempered distributions whose twisted product can be formed, see [11]. The motivation for this extension is obvious because it is desirable to include as many physical observables in the formalism as possible. The analysis performed here allows constructing larger $\star$-algebras of generalized functions including ultradistributions and hyperfunctions. This construction will be detailed in a subsequent paper.

Appendix

The following simple lemma is useful in examining product (6) and in finding the conditions under which series (2) converges in the spaces S^β_α.

Lemma. *If the space $S^\beta_\alpha(\mathbb{R})$ is nontrivial, then it contains a function f such that*

$$|\partial^n f(0)| \geq n^{\beta n} \tag{A1}$$

for all $n = 0, 1, 2, \ldots$, and the space $S^\alpha_\beta(\mathbb{R})$ contains an even nonnegative function $\hat{g}$ satisfying the inequality

$$\hat{g}(s) \geq e^{-|s|^{1/\beta}}. \tag{A2}$$

Proof. We note that the first statement of the lemma follows from the second. Indeed, let $g = \mathcal{F}^{-1}(\hat{g})$. Clearly, $\partial^n g(0) = 0$ for every odd n, and

$$|\partial^n g(0)| = \frac{1}{2\pi}\int s^n \hat{g}(s)ds \geq \frac{1}{2\pi}\int s^n e^{-|s|^{1/\beta}}ds$$

for every even n. The maximum of the last integrand occurs when $s = (\beta n)^\beta$. We let s_n denote this number. If $\beta > 1$, then the function $|s|^{1/\beta}$ is subadditive, and we have the inequalities

$$\int_{s_n}^{s_n+1} s^n e^{-|s|^{1/\beta}}ds \geq (s_n+1)^n e^{-(s_n+1)^{1/\beta}} \geq (\beta n)^{\beta n} e^{-\beta n - 1}.$$

A function $f(t)$ with property (A1) is obtainable from $g(t) + g'(t)$ by an appropriate scaling transformation. If $0 < \beta < 1$, then we use the inequality $|s+\sigma|^{1/\beta} \leq 2^{1/\beta}\big(|s|^{1/\beta} + |\sigma|^{1/\beta}\big)$ instead of subadditivity, which slightly complicates the formulas but yields the same result.

We now show that there exists a function $\hat{g} \in S^\alpha_\beta$ satisfying condition ($A2$). For simplicity, let $\beta > 1$ as before. We use the fact that the space S^α_β is an algebra under (ordinary) multiplication and is translation and dilation invariant. Starting from any nontrivial element in it and applying these operations, we can construct an even nonnegative function ω that also belongs to S^α_β and has the properties

$$|\partial^\kappa \omega(s)| \leq CA^\kappa \kappa^{\alpha\kappa} e^{-2|s|^{1/\beta}}, \qquad \int_{-1}^{+1} \omega(s)ds = e,$$

where $C, A > 0$ are sufficiently large constants. We set

$$\hat{g}(s) = \int e^{-|s-\sigma|^{1/\beta}} \omega(\sigma)d\sigma.$$

Clearly, $\hat{g}$ is also an even nonnegative function and belongs to S^α_β. Indeed, using the subadditivity of $|s|^{1/\beta}$, we obtain

$$|\partial^\kappa \hat{g}(s)| \leq \int e^{-|\sigma|^{1/\beta}} |\partial^\kappa \omega(s-\sigma)|\, d\sigma \leq C' A^\kappa \kappa^{\alpha\kappa} e^{-|s|^{1/\beta}},$$

where $C' = C\int e^{-|\sigma|^{1/\beta}}d\sigma$. Furthermore, $\hat{g}(s)$ satisfies the lower bound

$$\hat{g}(s) \geq e^{-(|s|+1|)^{1/\beta}} \int_{-1}^{+1} \omega(\sigma)d\sigma \geq e^{-|s|^{1/\beta}}.$$

This completes the proof. □

Acknowledgements. This paper was supported by the Russian Foundation for Basic Research (Grant No. 05-01-01049) and the Program for Supporting Leading Scientific Schools (Grant No. LSS-4401.2006.2).

References

[1] R. J. Szabo, *Quantum field theory on noncommutative spaces*, Phys. Rep. **378**, 207-299 (2003) [arXiv:hep-th/0109162].

[2] L. Álvarez-Gaumé and M. A. Vázquez-Mozo, *General properties of noncommutative field theories*, Nucl. Phys. B**668**, 293-321 (2003) [arXiv:hep-th/0305093].

[3] M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A.Tureanu, and Yu.A.Vernov, *Towards an axiomatic formulation of noncommutative field theories*, arXiv:hep-th/0402212.

[4] G. Fiore and J. Wess, *On "full" twisted Poincaré symmetry and QFT on Moyal-Weyl spaces*, Phys. Rev. D**75**, 105022 (2007) [arXiv:hep-th/0701078].

[5] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, New York, (1964).

[6] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak, and I. T. Todorov, *General Principles of Quantum Field Theory*, Kluwer, Dordrecht (1990).

[7] A. S. Wightman, *The choice of test functions in quantum field theory*, Adv. Math. Suppl. Stud., **7**B, 769-791 (1981).

[8] N. Ishibashi, S. Iso, H. Kawai, and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, Nucl. Phys. B**573**, 573-593 (2000) [arXiv:hep-th/9910004].

[9] D. J. Gross, A. Hashimoto, and N. Itzhaki, *Observables of non-commutative gauge theories*, Adv. Theor. Math. Phys. **4**, 893-928 (2000) [arXiv:hep-th/0008075].

[10] M. A. Soloviev, *Axiomatic formulations of nonlocal and noncommutative field theories*, Theor. Math. Phys. **147**, 660-669 (2006) [arXiv: hep-th/0605249].

[11] J. M. Gracia-Bondia and J. C. Várilly, *Algebras of distributions suitable for phase-space quantun mechanics. I*, J. Math. Phys. **29**, 869-879 (1988).

[12] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, Vol. 2, Acad. Press, New York (1968).

[13] B. S. Mityagin, *Nuclearity and other properies of the spaces of type S*, Russ. Math. Surv. **9**, 317-328 (1960) [in Russian].

[14] M. A. Evgrafov, *Asymptotic Estimates and Entire Functions*, Gordon and Breach, New York, (1962).

[15] V. Ya. Fainberg and M. A. Soloviev, *How can local properties be described in field theories without strict locality?* Ann. Phys. **113**, 421-447 (1978).

[16] G. V. Efimov, *Problems in Quantum Theory of Nonlocal Interactions*, Nauka, Moscow (1985) [in Russian].

[17] M. A. Soloviev, *PCT, spin and statistics, and analytic wave front set*, Theor. Math. Phys. **121**, 1377-1396 (1999) [arXiv:hep-th/0605243].

[18] J. W. Moffat, *Noncommutative and non-anticommutative quantum field theory*, Phys. Lett. B**506**, 193-199 (2001) [arXiv:hep-th/0011035].

Lebedev Physical Institute, Russian Academy of Sciences, Leninsky Prospect 53, Moscow 119991, Russia

E-mail address: `soloviev@lpi.ru`